Tutorial 1

Saturday, 6 August 2022 2:01 PM

- Why do we study measure theoretic probability?

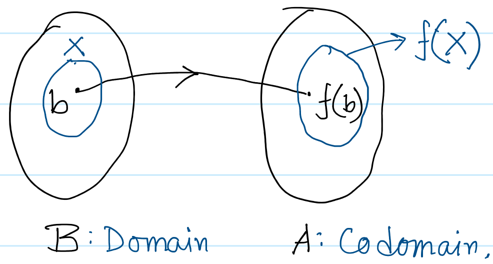
→ Because it is not easy to assign probabilities on the events from 'large' sample spaces.

→ One may find non-unique probability assignment to a single problem in non-mathematical terms.
ex- Bertrand Paradox.

- Functions.

Defⁿ (Function). Let A, B be sets. A function $f: B \to A$ is a rule that associates each element of B to a unique element of A.

- $f(b)$ is image of b, b is preimage of $f(b)$
- $f(X) = \{ f(b) : b \in X \}$ is image of X.



Range: $f(B) = \{ f(b) : b \in B \}$

- Not necessarily $f(B) = A$. If $f(B) = A$, then f is surjective.
- If each $a \in f(B)$ has unique preimage in B, then f is injective.
- If f is both injective and surjective, f is bijective.

- For every $a \in f(B)$ we can denote its preimage as $f^{-1}(a)$ such that $f(f^{-1}(a)) = a$

· Mind that $f^{-1}(a)$ is a set. If f is injective, $f^{-1}(a)$ is singleton set $\forall\, a \in f(B)$ and then only we can present $f^{-1}: f(B) \to B$ as a function. Still it is not exact inverse of $f: B \to A$. Surjection of f completes that.

· Cardinality:

· Two set are said to be equicardinal if $\exists$ a bijection between them. Let B is a set.

- B is countably infinite if $|B| = |\mathbb{N}|$
- B is finite if $|B| < |\mathbb{N}|$
- B is countable if $|B| \leq |\mathbb{N}|$

Ex 1. Let $(E_i : i \in \mathbb{N})$ be family of countably infinite sets. Then $\bigcup_{i\in\mathbb{N}} E_i$ is countable.

Proof: Let

$$E_1 = \{ e_{11}, e_{12}, e_{13} \dots \}$$
$$E_2 = \{ e_{21}, e_{22}, e_{23}, \dots \}$$
$$E_3 = \{ e_{31}, e_{32}, e_{33} \dots \}$$
$$\vdots$$

Let h :

$$\begin{array}{cccc} e_{11} & e_{12} & e_{13} & \dots \\ e_{21} & e_{22} & e_{23} & \dots \\ e_{31} & e_{32} & e_{33} & \dots \\ \vdots & & & \end{array} \qquad e_{ij} \to \frac{(i+j)(i+j-1)}{2} + i$$

Ex.2. Show that $[0,1]$ is uncountable.

Proof: Let us consider $[0,1]$ is countable. So we can so we should be able to enumerate $[0,1]$

Any element of $[0,1]$ can be written as infinite string of decimal digits i.e.

String of decimal digits i.e.

If $a \in [0,1]$ then, $a = 0.d_1 d_2 d_3 d_4 \dots$ where $d_i \in \{0,1,\dots 9\}$

$$\mathcal{D} = \{(d_1 d_2 d_3 d_4 \dots) : d_i \in \{0,1,\dots,9\}\}$$

Let us assume the $\mathcal{D}$ is countable. Then we can enumerate $\mathcal{D}$

Let us write elements of $\mathcal{D}$ in an arbitrary enumeration rule.

$$D_1 = d_{11}\ d_{12}\ d_{13}\ d_{14} \dots$$
$$D_2 = d_{21}\ d_{22}\ d_{23}\ d_{24} \dots$$
$$D_3 = d_{31}\ d_{32}\ d_{33}\ d_{44} \dots$$

Let $f : \{0,\dots,9\} \to \{0,\dots 9\}$ s.t.

$$f(n) = \begin{cases} 3 & \text{if } n \neq 3 \\ 4 & \text{if } n = 4. \end{cases}$$

$D' = f(d_{11})\ f(d_{22})\ f(d_{33})\ f(d_{44}) \dots$ is not any where in the list.

So, $\mathcal{D}$ is uncountable.

- Each dyadic rational is mapped to 2 elements in $\mathcal{D}$

eg $\frac{1}{2} \rightarrow 0.4999\dots$, $\frac{1}{2} \rightarrow 0.5000\dots$ We need to show that the set of dyadic rational is countable. Let that is $\mathcal{D}_n$

$$\mathcal{D}_n \cup (\mathcal{D} \setminus \mathcal{D}_n) = \mathcal{D} \text{ is uncountable.}$$

$\mathcal{D}_n$ is countable then $\mathcal{D} \setminus \mathcal{D}_n$ must be uncountable. which has a bijection with $[0,1]$. $\boxtimes$

- Measure Theoretic Probability - Construction of sets on which we can assign probability.

· Sample Space : Set of all outcomes of a random experiment.

Denote by Ω (Universal set in set theory terminology)

· Subsets of Ω are called events (not all)

· Def^n (Algebra). A collection $\mathcal{F}_0$ (set of sets) of subsets of Ω is called an algebra if.

1. $\Omega \in \mathcal{F}_0$
2. If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$
3. If $A \in \mathcal{F}_0$, $B \in \mathcal{F}_0$ then $A \cup B \in \mathcal{F}_0$.

· We can show $\bigcup_{i=1}^{n} A_i \in \mathcal{F}_0$, $\bigcap_{i=1}^{n} A_i \in \mathcal{F}_0$ if $\{A_i\}_{i=1}^{n} \subseteq \mathcal{F}_0$

This is less than what we need. (will come back to it).

Def^n (σ-Algebra). A collection $\mathcal{F}$ of subsets of Ω is called an algebra if.

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$

* For finite sample spaces all algebras are σ-algebra.

Ex.3. If $(A_n : n \in \mathbb{N})$ are subsets of $\mathcal{F}$ then

1. $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}_i$, 2. $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$ 3. $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$

Proof 1. $A_i \in \mathcal{F} \Rightarrow A_i^c \in \mathcal{F}$

$$\therefore \bigcup_{i \in \mathbb{N}} A_i^c \in \mathcal{F} \Rightarrow \left(\bigcap_{i \in \mathbb{N}} A_i\right)^c \in \mathcal{F}$$

$$\Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$$

2. We know that if $E_i \in \mathcal{F}$ $\forall$ $i \in \mathbb{N}$ then $\bigcup_{i \in \mathbb{N}} E_i \in \mathcal{F}$.

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We also know, $\phi \in \mathcal{F}$. Let $E_i = A_i$ for $i \in [n]$

and $E_i = \phi$ $\forall$ $i > n$

$$\therefore \bigcup_{i\in\mathbb{N}} E_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i>n} \phi\right) = \bigcup_{i=1}^{n} A_i.$$

3. You can do using similar technique.

* Any σ-algebra is algebra.

· Measure

· The concept of σ-algebra is used in measure theory, to set are to be assigned a measure. So we call $(\Omega, \mathcal{F})$ a measurable space and $A \in \mathcal{F}$ as $\mathcal{F}$-measurable set.

Def^n: A measure is a function $\mu : \mathcal{F} \to [0, \infty]$ s.t.

1. $\mu(\phi) = 0$
2. If $(A_n : n \in \mathbb{N})$ is a collection of disjoint $\mathcal{F}$-measurable sets then
$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

· $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Def^n A probability measure P on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0,1]$ s.t.

1. $P(\phi) = 0$
2. If $(A_n : n \in \mathbb{N})$ is a collection of disjoint $\mathcal{F}$-measurable sets then
$$P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} P(A_i)$$
3. $P(\Omega) = 1$.

· We call $(\Omega, \mathcal{F}, P)$ a probability space.

Ex 4. If $A_i \in \mathcal{F}$ $\forall i \in [n]$ for some $n \in \mathbb{N}$. then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \dots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

Proof: We will prove using mathematatic induction.

Base case: $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$

Induction hypothesis: $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \dots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$

Induction step:

$$\mathbb{P}\left(\bigcup_{i=1}^{n+1} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right)$$

$$= \sum_{i=1}^{n+1} \mathbb{P}(A_i) - \sum_{\substack{i,j=1 \\ i<j}}^{n} \mathbb{P}(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} \mathbb{P}(A_i \cap A_j \cap A_k) + \dots +$$

$$(-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) - \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{n+1}) + \sum_{\substack{i=1 \\ i<j}}^{n} \mathbb{P}(A_i \cap A_j \cap A_{n+1})$$

$$+ (-1)^{n-1} \sum_{j=1}^{n} \mathbb{P}\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{n} (A_i \cap A_{n+1})\right) \qquad + (-1)^{n} \mathbb{P}\left(\bigcap_{i=1}^{n+1} A_i\right)$$

Assigning measures to Uncountable sample spaces:

- Let us assign a uniform measure on the sample space $\Omega = [0, 1]$

Can we assign probabilities to every set in 2^{Ω}?

Ans No.

1. If we assign non-zero probability to each singleton, Probability of infinite sets will be unbounded.
2. If we assign zero probability to each singleton, probability of

2. If we assign zero probability to each singleton, probability of any interval is undefined.

Generated σ-algebra: σ-algebra generated by a collection of subsets $\mathcal{C}$ of Ω is the smallest σ-algebra that contains $\mathcal{C}$.

Ex 5. If $\mathcal{C}$ is an arbitrary collection of subsets of Ω and $\mathcal{H}$ is any σ-algebra such that $\mathcal{C} \subseteq \mathcal{H}$ then there exists σ-algebra $\sigma(\mathcal{C}) \subseteq \mathcal{H}$.

Proof: Let $\{\mathcal{F}_\alpha : \alpha \in A\}$ denote the collection of all σ-algebras that contain $\mathcal{C}$.

$\{\mathcal{F}_\alpha : \alpha \in A\}$ is not empty as $2^\Omega \in \{\mathcal{F}_\alpha : \alpha \in A\}$

Then $\mathcal{C} \subseteq \bigcap_{\alpha \in A} \mathcal{F}_\alpha$ is a σ-algebra. (Need to show)

Also if $\mathcal{C} \subseteq \mathcal{H}$ which is a σ-algebra then $\mathcal{H} \in \{\mathcal{F}_\alpha : \alpha \in A\}$ and $\bigcap_{\alpha \in A} \mathcal{F}_\alpha \subseteq \mathcal{H}$. This is true for any $\mathcal{H}$.

Ex. 6. $(\mathcal{F}_i,\ i \in \mathcal{I})$ is a collection of σ-algebras. Prove that $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-algebra.

Proof: $\phi \in \bigcap_{i \in I} \mathcal{F}_i$ as $\phi \in \mathcal{F}_i \ \forall i \in I$.

Let $A \in \bigcap_{i \in I} \mathcal{F}_i \Rightarrow A \in \mathcal{F}_i \ \forall i \in I$

$\Rightarrow A^c \in \mathcal{F}_i \ \forall \ i \in I$

$\Rightarrow A^c \in \bigcap_{i \in I} \mathcal{F}_i$

Let $A_n \in \bigcap_{i \in I} \mathcal{F}_i \quad \forall n \in \mathbb{N}. \Rightarrow A_n \in \mathcal{F}_i \ \forall i \in I, \forall n \in \mathbb{N}$

$\Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_i \ \forall \ i \in I$

$\Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{i \in I} \mathcal{F}_i$

So we showed all three condition of σ-algebra for $\bigcap_{i\in I} \mathcal{F}_i$. $\square$

Borel σ-algebra:

$\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by collection of half closed intervals. Let $\mathcal{C} := \{(-\infty, x] : x \in \mathbb{R}\}$, then

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}).$$

Ex. 6. Singleton sets are Borel sets.

Proof: For arbitrary b, $(-\infty, b] \in \mathcal{B}(\mathbb{R})$

Then $(-\infty, b-\frac{1}{n}] \in \mathcal{B}(\mathbb{R})$

$\Rightarrow (b-\frac{1}{n}, \infty) \in \mathcal{B}(\mathbb{R})$

$\Rightarrow \bigcap_{n\in\mathbb{N}} (b-\frac{1}{n}, \infty) \in \mathcal{B}(\mathbb{R}) \Rightarrow [b, \infty) \in \mathcal{B}(\mathbb{R})$

$\therefore (-\infty, b] \cap [b, \infty) = \{b\} \in \mathcal{B}(\mathbb{R})$

- ## Limit of Sets

Definition: Let $(A_n : n \in \mathbb{N})$ be a sequence of non-decreasing sets.

$$\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$$

If $(A_n : n\in\mathbb{N})$ is a sequence of non-increasing sets

$$\lim_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n$$

Definition: Let $(A_n : n\in\mathbb{N})$ be a sequence of sets

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m$$

Lemma: For a sequence of sets $(A_n : n \in \mathbb{N})$, we have

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

Proof: Let $E_n = \bigcup_{m \geq n} A_m$ $\therefore$ $(E_n : n \in \mathbb{N})$ is sequence of decreasing sets.

Let $F_n = \bigcap_{m \geq n} A_m$ $\therefore$ $(F_n : n \in \mathbb{N})$ is sequence of increasing sets.

Also, $F_1, F_2, \dots F_n \subseteq A_n$ and $F_m \subseteq A_m$ $\forall\ m \geq n$.

$$\therefore \bigcup_{i \in \mathbb{N}} F_i \subseteq \bigcup_{m \geq n} A_m \quad \forall\ n \in \mathbb{N}$$

$$\Rightarrow \bigcup_{i \in \mathbb{N}} F_i \subseteq \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m$$

Continuity of Probability: Let $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of events in $\mathcal{F}$. such that $\lim_{n\to\infty} A_n$ exists. We have

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$$

Proof:

$E_n = \bigcup_{m \geq n} A_m \in \mathcal{F} : n \in \mathbb{N}$ $\qquad$ $F_n = \bigcap_{m \geq n} A_m \in \mathcal{F} : n \in \mathbb{N}$

↳ non-increasing set $\qquad$ ↳ non decresing set.

$A_m \subseteq E_n \quad \forall\ m \geq n$ $\qquad$ $F_n \subseteq A_m \quad \forall\ m \geq n$

$P(E_n) \geq \sup_{m \geq n} P(A_m)$ $\qquad$ $P(F_n) \leq \inf_{m \geq n} P(A_m)$

$$\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right) = P\left(\bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m\right) \geq \limsup_n \mathbb{P}(A_n)$$

$$\lim_{n\to\infty} P(F_n) = P\left(\lim_{n\to\infty} F_n\right) = P\left(\bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_m\right) \leq \liminf_n \mathbb{P}(A_n)$$

$\limsup P(A_n) \leq P(\lim A_n) \leq \liminf P(A_n)$ (lim sup $P(A_n)$

$n \to \infty$

$$\limsup P(A_m) \leq P\left(\lim_{n\to\infty} A_n\right) \leq \liminf P(A_m)$$

$$\left(\lim_{n\to\infty} \sup_{m\geq n} P(A_m) = \limsup_{n\to\infty} P(A_n) \text{ and } \lim_{n\to\infty} \inf_{m\geq n} P(A_m) = \liminf_{n\to\infty} P(A_n). \right)$$

Also, $\liminf P(A_n) \leq \limsup (A_n)$

$$\Rightarrow \limsup_n P(A_n) = \liminf_n P(A_n) = \lim_n P(A_n)$$

$$\lim_n P(A_n) = P\left(\lim_n A_n\right) \quad \square$$