Tutorial 11

Monday, 26 September 2022 9:20 AM

Exercise 1 : Given $x_0 = i$, let $R_i = K$ if $X_1 = i, \dots, X_{K-1} = i$, $X_K \neq i$; i.e. R_i is the exit time from state i.

a. Compute $P(R_i = n \mid X_0 = i)$ $\forall n \in \mathbb{N}$. Then compute $\mathbb{E}[R_i \mid X_0 = i]$

b. If the DTMC has only two states, 0 and 1, with $X_0 = 0$, find the mean time until the DTMC first exits 1 and enters 0. (Assume, $P_{00}, P_{01}, P_{10}, P_{11} \in (0,1)$)

Sol^n : a. $P(R_i = n \mid X_0 = i) = P(X_n \neq i, X_{n-1} = i, \dots, X_1 = i \mid X_0 = i)$

$$= P(X_n \neq i \mid X_{n-1} = i)\, P(X_{n-1} = i \mid X_{n-2} = i) \cdots$$
$$\cdots P(X_1 = i \mid X_0 = i)$$

$$= (1 - P_{ii})\, P_{ii}^{\,n-1} \qquad n \in \mathbb{N}$$

$$\therefore \quad \mathbb{E}[R_i \mid X_0 = i] = \sum_{n \in \mathbb{N}} n\,(1 - P_{ii}) \cdot P_{ii}^{\,n-1}$$

$$= \frac{1}{1 - P_{ii}}$$

b. Let T be the time when the DTMC first exits 1 to enter 0.

$$\mathbb{E}[T \mid X_0 = 0] = \mathbb{E}[R_0 \mid X_0 = 0] + \mathbb{E}[R_1 \mid X_1 = 1]$$

$$= \frac{1}{1 - P_{00}} + \frac{1}{1 - P_{11}}$$

First Passage time Distributions & Recurrence :

- k^{th} hitting time : $\tau_k^{ij} = \inf\{n > \tau_{k-1} : X_n = j \text{ s.t } X_0 = 0\}$
- 1^{st} hitting time $\tau_1^{ij} = \inf\{n : X_n = j \text{ s.t. } X_0 = 0\}$

- $f_{ij}^{(n)} = P\{\tau_1^{ij} = n\}$
- $f_{ij} = P\{\tau_1^{ij} < \infty\} = \sum_{n \in \mathbb{N}} f_{ij}^{(n)}$ – probability that a state eventually hits j
- First passage time distribution : $\left(\left(f_{ij}^{(n)} : n \in \mathbb{N}\right), 1 - f_{ij}\right)$
- First recurrence time distribution : $\left(\left(f_{ii}^{(n)} : n \in \mathbb{N}\right) 1 - f_{ii}\right)$
- A state i is recurrent if $f_{ii} = 1$ else i is transient.
- Mean recurrence time : $\mu_i = \mathbb{E}_i[\tau_1^{ii}]$
- Positive recurrent : $\mu_i < \infty$

  Null recurrent : $\mu_i = \infty$

Example of such distributions:

Let the first recurrent time distribution is as follows

$$P(\tau_1^{ii} = n) = \frac{6}{\pi^2} \cdot \frac{1}{n^2} \qquad n \in \mathbb{N}$$

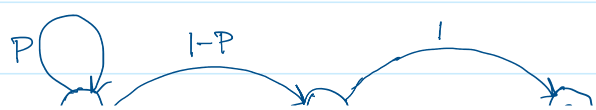
Then, $\mu_i = \mathbb{E}_i[\tau_1^{ii}] = \frac{6}{\pi^2} \sum_{n \in \mathbb{N}} n \cdot \frac{1}{n^2} = \infty$

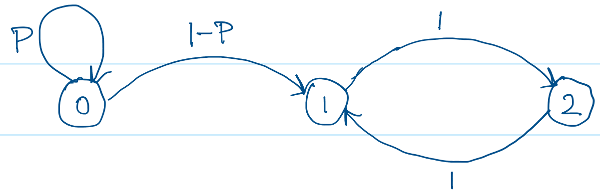
Alternatively, if the first recurrent time distribution is

$$P(\tau_1^{ii} = n) = \frac{1}{\zeta(3)} \cdot \frac{1}{n^3} \qquad n \in \mathbb{N}$$

Then, $\mu_i = \mathbb{E}_i[\tau_1^{ii}] = \frac{1}{\zeta(3)} \cdot \sum_{n \in \mathbb{N}} n \cdot \frac{1}{n^3} = \frac{1}{\zeta(3)} \cdot \frac{\pi^2}{6} < \infty$

Exercise 2 : Consider a DTMC with the following transition graph





Find for each state, whether it is transient or recurrent. If a state is recurrent, find if it is positive recurrent or null recurrent. Assume $p < 1$.

Sol^n: $f_{00}^{(1)} = p \quad f_{00}^{(n)} = 0 \quad \forall n \in \mathbb{N} \setminus \{1\}. \quad \Rightarrow \quad f_{00} = p.$

So, 0 is transient state

$$f_{11}^{(1)} = 0, \quad f_{11}^{(2)} = 1, \quad f_{11}^{(n)} = 0 \quad \forall n \in \mathbb{N} \setminus \{1, 2\}.$$

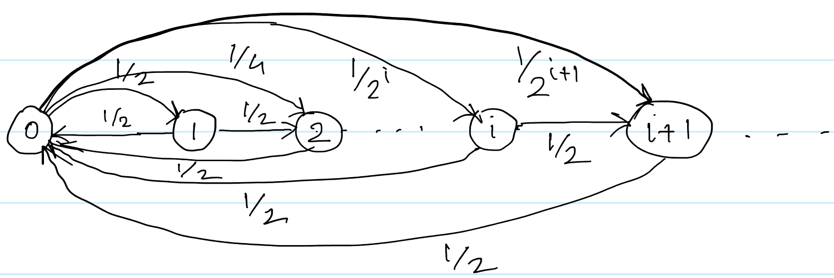
So, $f_{11} = 1$ and hence 1 is recurrent state.

$\mu_1 = \sum k \cdot f_{11}^{(k)} = 2 \cdot 1 = 2 < \infty$. So 1 is a positive recurrent state. Similarly, 2 is also a positive recurrent state.

Exercise 3: $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ is a DTMC on $\mathcal{X} = \mathbb{N} \cup \{0\}$ with $P_{0i} = \left(\frac{1}{2}\right)^i$ for $i \in \mathbb{N}$ and $P_{i0} = \frac{1}{2}$; $P_{i,i+1} = \frac{1}{2}$.

Compute $f_{00}^{(n)}$ for $n \geq 1$. Is 0 recurrent? If yes, is it positive recurrent?

Sol^n. The state transition diagram for the DTMC is



$$f_{00}^{(1)} = 0, \quad f_{00}^{(2)} = \frac{1}{2} \sum_{i \in \mathbb{N}} \frac{1}{2^i} = \frac{1}{2}.$$

$f^{(k)} = \frac{1}{\ldots} \sum \frac{1}{\ldots} = \frac{1}{\ldots}$ for $k \in \mathbb{N} \setminus \{\ldots\}$

$i \in \mathbb{N}$

$$f_{00}^{(k)} = \frac{1}{2^{k-1}} \sum_{i \in \mathbb{N}} \frac{1}{2^i} = \frac{1}{2^{k-1}} \quad \text{for } k \in \mathbb{N} \setminus \{0\}$$

Hence, $\sum_{n \in \mathbb{N}} f_{00}^{(n)} = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \quad \underline{1}$ i.e. 0 is recurrent.

$\mathbb{E}_0[\mu_0] = \sum_{n \in \mathbb{N}} (n+1) \frac{1}{2^n} = 3 < \infty$ hence 0 is positive recurrent.